Introduction to Mechanism Design Theory

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Assigning indivisible objects: top trading algorithm

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A brief introduction to Matching Theory

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- Suppose that there exists an initial assignment of objects to agents $\mu : N \longrightarrow O$, without loss of generality let assume that $\mu(i) = o_i$ for all $i = 1, ..., n$.
- Each agent $i \in N$ has a strict preference P_i over the set O.

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- Examples of problems with these characteristics are housing market (the original example in Shapley and Scarf (1974)) the problem of assigning students to dorms or more recently paired kidney exchange protocols, analyzed for the first time in the seminal work by Roth, Sönmez and Ünver (2004).

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- The possibility of living donation generate interesting new strategies to alleviate the (universal) shortage of kidneys.
- Two incompatible donor–patient pairs may be mutually compatible, and a swap of donors between the two pairs would result in two successful transplantations (Paired Kidney Exchange, PKE).

Preferences and Rules

- A profile $P = (P_1, ..., P_n)$ is a list of strict preferences over O, one for each agent. Given a strict preference P_i of agent i, we define the weak preference R_i on O in the following way: for all $j, j' = 1, ..., n$, $o_j R_i o_{j'}$ if either $o_j = o_{j'}$ or $o_j P_i o_{j'}$.

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- Fix N, O, μ and P, we define the quadruple (N, O, μ, P) an (assignment) problem.
- A solution of an assignment problem (N, O, μ, P) is an allocation rule $\alpha : N \to O$.
- In the example of PKE a solution α assigns to each patient a kidney (if $\alpha(i) = \mu(i) = o_i$ we interpret that patient i remains in dialysis since she does not get any compatible kidney.

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2. The final allocation must be efficient. An allocation rule $\alpha : N \longrightarrow O$ is efficient in the problem (N, O, P, μ) if it does not exists any other allocation rule $\nu : N \longrightarrow O$ such that for all $i \in N$, $\nu(i) R_i \alpha(i)$ and $\nu(j) P_j \alpha(j)$ for some $j \in N$.

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3. No subset of agents that can increase their welfare by re-assigning their own initial objects (according to the allocation μ) among themselves; in other words, the allocation cannot be blocked by any subset of agents.

Definition An allocation $\alpha : N \to O$ belongs to the core of the problem (N, O, P, μ) if there does not exist any blocking coalition of agents $S \subseteq N$ and allocation $\nu : N \to O$ such that:

- $\nu(i) \in \mu(S)$ for all $i \in S$,
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- Every allocation in the core is individually rational and efficient.
- Consider in fact in the previous definition the coalition formed by a unique agent $(S = \{i\})$ and the entire coalition $(S = N)$.
- Shapley and Scarf (1974) proved the fundamental result that the core of each assignment problem is not empty

Theorem

(Shapley and Scarf, 1974) *Every assignment problem has a non-empty core*

- The paper by Shapley and Scarf (1974) contains two proofs. One is an indirect and non-constructive proof and the other one, according to the authors suggested by David Gale, consists in defining an algorithm, currently known as the Gale top trading cycle (TTC).

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- At each round (i) a graph is constructed where the vertices are pairs ,*objects and agents*, and the agents are those who have still not got an object in the previous round (ii) an arrow goes from each agent to his most desirable object ; (iii) the vertices of the directed graph are identified and (iv) if they form a cycle, to each agent in the cycle is assigned his most preferred object.

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- If preferences are strict and n is finite in each round there is at least one cycle and if there are more than one, they do not intersect. Hence the algorithm assigns each object to some agent in a finite number of rounds. More formally,

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- Let $\eta : N \to O$ denote the assignment obtained by using the TTC algorithm for the problem (N, O, μ, P) and let K be the last round of the algorithm. The following example illustrates the TTC algorithm.

Example Let (N, O, μ, P) be an assignment problem with $|N| = |O| = 8$, $\mu(i) = o_i$ for each $i = 1, ..., 8$, and let P be the profile of agents' preferences represented in Table 1 where the object inside a square is the initial endowment μ of each agent.

Table 1

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
o_2	o_3	o_1	o_8	o_4	o_8	o_4	o_6
o_3	o_1	$\boxed{o_3}$	o_7	o_7	o_1	o_8	$\boxed{o_8}$
o_5	$\boxed{o_2}$	o_7	$\boxed{o_4}$	o_3	$\boxed{o_6}$	o_3	o_1
o_6	o_8	o_2	o_1	o_6	o_5	o_6	o_2
o_8	o_6	o_5	o_2	o_1	o_4	o_1	o_3
$\boxed{o_1}$	o_4	o_8	o_3	o_8	o_3	o_5	o_7
o_7	o_7	o_6	o_5	o_2	o_2	o_2	o_5
o_4	o_5	o_4	o_6	$\boxed{o_5}$	o_7	$\boxed{o_7}$	o_4

In the figure 2 we represent the three rounds of the algorithm to get the assignment μ.

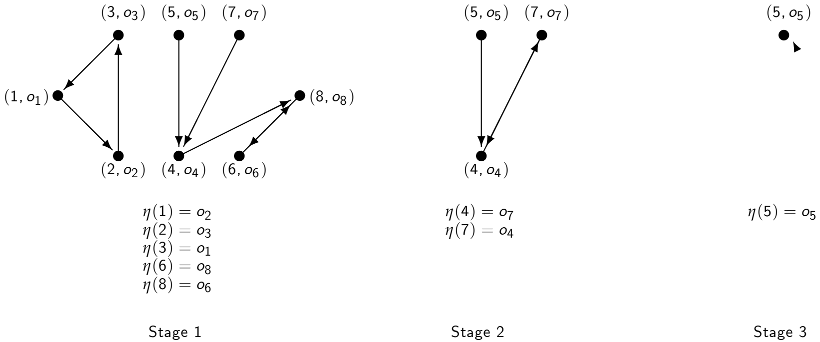

Figure 2

We prove now that the TTC algorithm selects an allocation in the core.

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- Given that, no agent in S_2 can be a member of a blocking coalition since each of them get his preferred object in $O \backslash \eta(S_1)$.

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- Given that, no agent in S_2 can be a member of a blocking coalition since each of them get his preferred object in $O \backslash \eta(S_1)$.
- Proceeding iteratively we get that η is an assignment in the core of the problem (N, O, μ, P) since it cannot be blocked by any coalition

Roth and Postlewaite (1977) show that there exists a unique allocation in the core, which is hence the allocation selected by the TTC algorithm.

Theorem

(Roth and Postlewaite, 1977) The core of each problem contains at most one allocation.

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- Let k be the first round of the algorithm TTC in which there exists an agent i in S_k (the set of agents who belong to some cycle in round k, that is to whom some object is assigned at round k) with the property that $\nu(i) \neq \eta(i)$; iif there is more than one, arbitrarily select one among them.

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- Therefore, $i \in S_k$ and for all j who get an object in some round before k according to η (that is for all $j \in S_1 \cup \ldots \cup S_{k-1}$), we have $\nu(j) = \eta(j)$.

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- Therefore, $i \in S_k$ and for all j who get an object in some round before k according to η (that is for all $j \in S_1 \cup \ldots \cup S_{k-1}$), we have $\nu(j) = \eta(j)$.
- Hence for all $j \in \cup_{l=1}^{k-1} S_l$, $\eta(j) R_j \nu(j)$ and $\eta(j) \in \mu(S_l)$ for some $l = 1, ..., k-1$. Moreover by definition of η, $\eta(i) P_i \nu(i)$ since $\nu(i) \neq \eta(i)$. It follows that the coalition $\cup_{l=1}^{k-1} S_l \cup \{i\}$ blocks the assignment ν. Therefore, ν is not in the core of (N, O, μ, P).

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- Fix N, O and μ and let A be the set of social alternatives;, that is $A = \{\alpha : N \to O \mid \alpha \text{ is bijective}\}$. In this case, each agent i is only interested in the object is assigned to him.
- His strict preferences are defined over the set of objects O (and not in A). Therefore, the set A_i of social alternatives which describes the characteristics of the alternatives which are of interest for agent i is defined, for all $\alpha \in A$, $[\alpha]_i = \{\beta \in A \mid \beta(i) = \alpha(i)\}$, and $[\alpha]_i$ represents the equivalence class which contains the assignment α.

- Given a strict preference P_i on O we can define, abusing notation, the weak preference R_i on the set of social alternatives A in the following way: for all pairs $\alpha, \alpha' \in A$, $\alpha R_i \alpha'$ if and only if, either $\alpha(i) = \alpha'(i)$ or $\alpha(i) P_i \alpha'(i)$.

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- The weak preference R_i on A has many indifferences since agent i is indifferent among all allocations where i gets the same object!
- Notice that the structure of the problem makes that the set of preferences over social alternatives is not the universal set, but a restricted domain of preferences, and therefore there is room for a positive result since one of the fundamental assumption of the Gibbard-Satterthwaite theorem is not satisfied.

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- As usual a social choice function $f : \mathcal{P}^n \rightarrow A$ is *manipulable* if there exists a profile $P = (P_1, ..., P_n) \in \mathcal{P}^n$, an agent $i \in N$ and a preference $P_i' \in \mathcal{P}$ such that

$$f_i(P_i', P_{-i}) P_i f_i(P_i, P_{-i});$$

that is, agent i gets a better object (according to his preference P_i) reporting P_i' instead of reporting P_i. Roth (1982a) proves that a social choice function that selects for each profile P the core of the problem (N, O, μ, P) is strategy-proof

Theorem

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Proof

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- Let $P \in \mathcal{P}^n$ be any arbitrary profile. Let $S_1, ..., S_K$ be the set of agents which form part of some cycles and to whom, applying the TTC algorithm in order to get $\eta = \varphi(P)$, are assigned objects in rounds $1, ..., K$; that is $i \in S_k$ means that agent i belong to a cycle at round k. The proof is by iteration in the cycles:

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$k \leq 1$ Each agent in S_1 gets in η his most preferred object according to P. Hence he cannot benefit by misreporting any preference different than P. Observe also that the cycles of S_1 are the same independently of the reports of agents $N \backslash S_1$.

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$k \geq 2$ Each agent in S_k (per a $k \geq 2$) in η receives his preferred object in the set of objects $O \backslash (\eta(S_1) \cup ... \cup \eta(S_{k-1}))$, according to P. Since any previous set $S_1 \cup ... \cup S_{k-1}$ is not affected if any agent in S_k reports a different preferences, agents in $S_1 \cup ... \cup S_{k-1}$ still continue to get the same objects. Therefore in applying the TTC algorithm, no agent in S_k can benefit by misreporting

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- However it may be still reasonable to require that the social choice function f is individually rational and efficient (for each profile $P \in \mathcal{P}^n$, $f(P)$is an individually rational and efficient allocation according to P), and that it provides the incentives to truthfully report the preferences.

- Ma (1994) shows that the intermediate coalitions ($S \neq \{i\}$ and $S \neq N$) have not any additional power in blocking allocations.

Theorem

(Ma, 1994) The social choice function $f : \mathcal{P}^n \rightarrow A$ is individually rational , efficient and strategy-proof if and only if it selects the core.

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- Suppose that at each object. o_i is assigned a price $p_{o_i} \geq 0$. We say that an object o_j is affordable for agent i in the price vector $p = (p_{o_1}, ..., p_{o_n}) \in \mathbb{R}^n_+$ if $p_{o_j} \leq p_{\mu(i)}$; that is if i can buy the object $o_j$at price p_{o_j} after having sold $\mu(i)$ at price p_{o_i}.

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- An assignment $\nu : N \to O$ is an *equilibrium* of the problem (N, O, μ, P) if there exists a vector of prices $p = (p_{o_1}, ..., p_{o_n})$ such that for each agent i, $\nu(i)$ is the most preferred object by the agent i among the affordable objects for him at $p = (p_{o_1}, ..., p_{o_n})$.

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- Roth and Postlewaite (1977) show that all allocation problems have a unique equilibrium and it coincides with the assignment in the core.

Theorem

(Roth and Postlewaite, 1977) For each assignment problem there exists a unique assignment which is an equilibrium and coincides with the core.